Lower bounds for Morse index of constant mean curvature tori

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Abstract: We give three lower bounds for the Morse index of a constant mean curvature torus in Euclidean 3-space in terms of its spectral genus g. The first two lower bounds grow linearly in g and are stronger for smaller values of g, while the third grows quadratically in g but is weaker for smaller values of g.

1. Introduction

The Morse index of a complete constant mean curvature (CMC) H surface in $\boldsymbol{R}^3$ with $H \neq 0$ is finite if and only if the surface is compact [23] [16], and is 0 (i.e. the surface is stable) if and only if the surface has genus 0 and hence is a round sphere [4]. It is also understood how to find all CMC tori [6] [18]. Thus, to search for the least possible index of unstable CMC surfaces, it is natural to begin with compact CMC tori. The simplest of them, the original Wente tori [26] [1] [25] with spectral genus $g = 2$, have index ≥ 8 [15] [21] [22], suggesting that perhaps no unstable CMC surface has index less than 8. In this direction, we show here that CMC tori with large g must have large index. (CMC tori exist for every $g \geq 2$ [17] [10].)

2. Description of CMC 1 tori

Any CMC torus can be described as a conformal isometric immersion

$$F : \boldsymbol{C}/\Lambda \to \boldsymbol{R}^3 ,$$

where Λ is a lattice in the complex plane $\boldsymbol{C}$, and the induced Riemannian metric on $\boldsymbol{C}/\Lambda$ is

$$ds^2 = e^u \cdot ds^2_{Eucl} , \qquad \text{where} \quad ds^2_{Eucl} = dx^2 + dy^2$$

is the standard Euclidean metric, and $u(z := x + iy) : \boldsymbol{C}/\widetilde{\Lambda} \to \boldsymbol{R}$ is doubly periodic with respect to another lattice $\widetilde{\Lambda}$ of $\boldsymbol{C}$. As CMC tori have no umbilic points [6], we may further assume the mean curvature and Hopf differential are

$$H = 1 \quad \text{and} \quad Q := \langle F_{zz}, \vec{N} \rangle = 1/2 ,$$

where $\vec{N}$ is a unit normal vector to the surface, and hence u satisfies the sinh-Gordon equation

$$\partial_z \partial_{\bar{z}} u + \sinh u = 0 .$$

Furthermore, u is smooth, i.e. $u \in C^\infty(\boldsymbol{C}/\widetilde{\Lambda})$. (The above facts are explained in more detail in any of [6] [15] [21] [22] [25] [26].)

Let Π (resp. $\widetilde{\Pi}$) represent a fundamental domain of the lattice Λ (resp. $\widetilde{\Lambda}$). Suppose that m copies of $\widetilde{\Pi}$ translated by vectors in $\widetilde{\Lambda}$ can be placed within Π with disjoint interiors. In other words, we have at least m disjoint congruent open regions on the torus F, each representing a region of double periodicity for u. For CMC tori with many symmetries, m can be large; for example, the original Wente tori can have arbitrarily large m. Since at the very least we may take $\Lambda = \widetilde{\Lambda}$ and $\Pi = \widetilde{\Pi}$, we may assume

$$m \geq 1 .$$

The function u can be described with theta functions (Theorems 7.2 and 8.1 of [6]):

$$u(z) = 2 \log \left(\frac{\theta(i\mathrm{Re}(Uz) + D + i\pi(1, 1, ..., 1))}{\theta(i\mathrm{Re}(Uz) + D)} \right) ,$$

where the theta function θ (as defined in [6]) is determined by a spectral curve of genus $g \geq 2$ defined in [6], and $D \in i\boldsymbol{R}^g$ is arbitrary, and U is defined in Theorem 7.1 of [6]. As the choice of D does not affect the periodicity of the surface, D gives a smooth $g - 2$ parameter family of CMC tori. Furthermore, there is a heirarchy of solutions $v_j : \boldsymbol{C}/\widetilde{\Lambda} \to \boldsymbol{R}$ to the linearized sinh-Gordon equation

$$L(v_j) = 0 , \qquad L := -\partial_z \partial_{\bar{z}} - \cosh u \tag{1}$$

given recursively by the following procedure [7]: with the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

as given in [6], define off-diagonal matrices R_j recursively by

$$R_1 = -\frac{1}{2} u_z \sigma_2 \ , \quad R_2 = \frac{1}{2} u_{zz} \sigma_1 \ , \quad i\,[R_{k+1}, \sigma_3] = -u_z \sum_{n=1}^{k-1} R_n \sigma_1 R_{k-n} - 2\partial_z R_k \ , \quad k \geq 2 \ .$$

Now define K_j recursively by

$$K_1 = -i\sigma_3 \ , \quad K_2 = -u_z \sigma_1 \ , \quad K_{j+1} = -i\,[R_j, \sigma_3] - \sum_{i=2}^{j} K_i R_{j+1-i} \ , \quad j \geq 2 \ .$$

For all positive even $j \in 2 \cdot \boldsymbol{Z}^+$, we find that $K_j = \rho_j \sigma_1$ for some scalar function ρ_j. For example, the first three ρ_j are

$$\rho_2 = -\partial_z u \ ,$$
$$\rho_4 = -\frac{1}{2}(\partial_z u)^3 + \partial_z^{(3)} u \ ,$$
$$\rho_6 = -\frac{3}{8}(\partial_z u)^5 + \frac{5}{2}\partial_z u (\partial_z^{(2)} u)^2 + \frac{5}{2}(\partial_z u)^2 \partial_z^{(3)} u - \partial_z^{(5)} u \ ,$$

where $\partial_z^{(n)}$ represents the n'th derivative with respect to z. Let

$$v_j := \mathrm{Re}(\rho_{j+1}) \text{ for } j \text{ odd}, \qquad v_j := \mathrm{Im}(\rho_j) \text{ for } j \text{ even}. \tag{2}$$

It is proven (with slightly differing notation) in Proposition 3.1 of [18] that these v_j satisfy equation (1).

As L is elliptic and the v_j are defined on the compact space $\boldsymbol{C}/\widetilde{\Lambda}$, only finitely many v_j can be linearly independent; but if the spectral genus of the torus is g and the spectral curve is nonsingular in the sense of [6], then at least the first $g-1$ functions $v_1, v_2, ..., v_{g-1}$ are linearly independent. We shall assume that the spectral curve is nonsingular, as there is a nonrigorous argument in [6] to show that the singular case never occurs.

Note that we can also consider the v_j to be defined on $\boldsymbol{C}/\Lambda$ as well as on $\boldsymbol{C}/\widetilde{\Lambda}$, since u is well-defined on $\boldsymbol{C}/\Lambda$ as well as on $\boldsymbol{C}/\widetilde{\Lambda}$.

3. Definition of Morse index

We now turn to the definition of Morse index. Let

$$F(t) : \boldsymbol{C}/\Lambda \to \boldsymbol{R}^3 \ , \quad t \in (-\epsilon, \epsilon) \ , \quad F(0) = F$$

be a smooth variation of F through immersions $F(t)$. Let $\vec{E}(t)$ be the variation vector field on $F(t)$. We can assume, by reparametrizing $F(t)$ for nonzero t, that $\vec{E}(0) = v\vec{N}$, $v \in C^\infty(\boldsymbol{C}/\Lambda)$. Let $a(t)$ =area$(F(t))$. The first variational formula is

$$a'(0) := \left.\frac{d}{dt} a(t)\right|_{t=0} = -\int_{\boldsymbol{C}/\Lambda} v dA \ ,$$

where $dA = e^u dxdy$. Let $V(t)$ =volume$(F(t))$, as defined in [4]. Then $V'(0) = \int_{\boldsymbol{C}/\Lambda} v dA$. The variation is *volume-preserving* if $\int_{\boldsymbol{C}/\Lambda} \langle \vec{E}(t), \vec{N}(t) \rangle dA(t) = 0$ for all $t \in (-\epsilon, \epsilon)$. In particular, $\int_{\boldsymbol{C}/\Lambda} v dA = 0$ when $t = 0$, so $a'(0) = 0$ and F is critical for area amongst all volume-preserving variations.

So to see which volume-preserving variations reduce area, one must consider which of them make the following second variation formula (for volume-preserving variations) negative:

$$a''(0) := \left.\frac{d^2}{dt^2} a(t)\right|_{t=0} = \int_{\boldsymbol{C}/\Lambda} \{|\nabla v|^2 - (4H^2 - 2K)v^2\} dA = 4\int_{\boldsymbol{C}/\Lambda} v L v dxdy \ , \tag{3}$$

where K and ∇ are the Gaussian curvature and gradient with respect to ds^2, and L is as in (1). (Note that actually $H = 1$ here.)

Definition 1. The *index* $\mathrm{Ind}(F)$ is the maximum possible dimension of a subspace $\mathcal{U} \subseteq C^\infty(\boldsymbol{C}/\Lambda)$ for which $\int_{\boldsymbol{C}/\Lambda} v dA = 0$ and $\int_{\boldsymbol{C}/\Lambda} v L v dx dy < 0$ for all nonzero $v \in \mathcal{U}$.

Let $L^2(\boldsymbol{C}/\Lambda)$ be the Hilbert space of measurable functions with finite L^2 norm, where the standard L^2 inner product $\langle \cdot, \cdot \rangle_{L^2}$ on $L^2(\boldsymbol{C}/\Lambda)$ is defined with respect to the metric ds^2_{Eucl}. The complete set of eigenvalues for L is discrete and can be listed as

$$\lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \nearrow +\infty$$

with associated eigenfunctions ν_j, i.e.

$$L\nu_j = \lambda_j \nu_j \ ,$$

where the $\nu_j \in C^\infty(\boldsymbol{C}/\Lambda)$ are chosen to form an orthonormal basis for $L^2(\boldsymbol{C}/\Lambda)$.

The next lemma is in [3] and [21] and other places as well, but we include a brief proof.

Lemma 1.

$$\mathcal{K} - 1 \leq \mathit{Ind}(F) \leq \mathcal{K} \ ,$$

where $\mathcal{K}$ is the number of strictly negative eigenvalues of L.

Proof. We have $\lambda_{\mathcal{K}} < 0 \leq \lambda_{\mathcal{K}+1}$. Since there exist solutions v_j as in (2) that solve $Lv_j = 0$, in fact $\lambda_{\mathcal{K}+1} = 0$.

Let $\mathcal{U} := \mathrm{span}\{\nu_1, \dots, \nu_{\mathcal{K}}\}$. For any nonzero $\nu = \sum_{i=1}^k a_i \nu_i \in \mathcal{U}$ for $a_1, ..., a_{\mathcal{K}} \in \boldsymbol{R}$, we have

$$\int_{\boldsymbol{C}/\Gamma} \nu L \nu dx dy = \sum_{i=1}^{\mathcal{K}} a_i^2 \lambda_i < 0 \ .$$

Therefore, if we choose $\hat{\mathcal{U}}$ to be a subspace of $C^\infty(\boldsymbol{C}/\Gamma)$ of maximum dimension such that $\int_{\boldsymbol{C}/\Gamma} \hat{\nu} L \hat{\nu} dx dy$ restricted to $\hat{\nu} \in \hat{\mathcal{U}}$ is negative definite, then $\dim(\hat{\mathcal{U}}) \geq \dim \mathcal{U} = \mathcal{K}$.

Suppose that $\dim(\hat{\mathcal{U}}) > \mathcal{K}$, and let $P : \mathcal{U} \longrightarrow \hat{\mathcal{U}}$ be the projection of $\mathcal{U}$ to $\hat{\mathcal{U}}$ with respect to the L^2 norm. Since $\dim(P(\mathcal{U})) \leq \mathcal{K}$ there exists a $\hat{\nu} \in \hat{\mathcal{U}}$ with $\langle \hat{\nu}, \hat{\nu} \rangle_{L^2} = 1$ such that $\hat{\nu} \perp_{L^2} P(\mathcal{U})$. It follows that $\hat{\nu} \perp_{L^2} \mathcal{U}$ and so $\int_{\boldsymbol{C}/\Gamma} \hat{\nu} L \hat{\nu} dx dy \geq 0$, a contradiction. Hence $\dim(\hat{\mathcal{U}}) = \mathcal{K}$ and $\mathrm{Ind}(F) \leq \mathcal{K}$.

Now, let $\tau : \hat{\mathcal{U}} \longrightarrow \boldsymbol{R}$ be the linear functional defined by

$$\tau(\hat{\nu}) = \int_{\boldsymbol{C}/\Gamma} \hat{\nu} dA \ .$$

Since the dimension of the kernel of τ is at least $\mathcal{K} - 1$, we have by Definition 1 that

$$\mathrm{Ind}(F) \geq \dim(\mathrm{Ker}(\tau)) \geq \mathcal{K} - 1 \ .$$

□

4. Two preliminary lemmas

Before stating and proving our main theorem, we give two lemmas needed for the proof. The first is a generalization of the Euler formula for graphs, which is classical and very well-known, but as we will need to allow somewhat nonstandard "graphs" that include closed loops, we give a proof.

Henceforth we shall refer to a compact connected Riemann surface without boundary as a *closed* Riemann surface.

Definition 2. Let M be a closed Riemann surface.

1) A finite embedded *graph-with-loops*

$$\mathcal{G} = \mathcal{G}' \cup \sum_{j=1}^{r} \gamma_j$$

is the union of an unoriented finite embedded graph $\mathcal{G}'$ in M with a finite number of disjoint closed loops $\gamma_1, ..., \gamma_r \subset M$ that do not intersect $\mathcal{G}'$. We allow $\mathcal{G}'$ to be disconnected, and we allow $\mathcal{G}'$ to have loop-edges. (Loop-edges are edges whose two endpoints are the same vertex, not to be confused with closed loops.) There are no vertices on the closed loops γ_j.

2) Let $\mathcal{F}$ denote the number of faces of $\mathcal{G}$, that is, suppose that $M \setminus \mathcal{G}$ consists of $\mathcal{F}$ components, which we call $V_1, ..., V_{\mathcal{F}}$. We must allow the possibility that some of the V_j are not homeomorphic to disks, as $\mathcal{G}$ can contain closed loops. In fact, some V_j might not even be homeomorphic to planar domains. Let the number of edges (resp. vertices) of $\mathcal{G}'$ be $\mathcal{E}'$ (resp. $\mathcal{V}'$). Counting each closed loop γ_j as one edge, we can say that $\mathcal{G}$ has $\mathcal{E} = \mathcal{E}' + r$ edges and $\mathcal{V} = \mathcal{V}'$ vertices.

Lemma 2. *Let $\mathcal{G} = \mathcal{G}' \cup \sum_{j=1}^{r} \gamma_j$ be a graph-with-loops on a closed Riemann surface M. Suppose that $\mathcal{G}'$ is not empty (i.e. $\mathcal{G}'$ has at least one vertex). Let $\chi(M) = 2 - 2 \cdot genus(M)$ be the Euler characteristic of M. Then, with $\mathcal{F}$ and $\mathcal{E}$ and $\mathcal{V}$ as in part 2 of Definition 2, we have*

$$\mathcal{F} - \mathcal{E} + \mathcal{V} \geq \chi(M) \ . \tag{4}$$

Remark. Strict inequality can occur in Equation (4). As a simple example, consider a graph on a torus that has one loop-edge e and one vertex p, where e lies in a homotopically trivial loop and both its ends connect to p. Then strict inequality will hold. This example is too simple to occur in the proof of Theorem 4, but it illustrates why we can only invoke the inequality (4) (and cannot assume equality) in that proof.

Remark. Lemma 2 does not hold without the assumption that $\mathcal{G}'$ is nonempty, and a simple counterexample is to let $\mathcal{G}$ consist of only a single closed loop γ_1 in the sphere S^2.

Remark. Equality can hold in Equation (4) even if some components V_j of $M \setminus \mathcal{G}$ are not homeomorphic to disks. For example, consider a graph $\mathcal{G} = \{e\} \cup \{p\} \cup \gamma_1$ on the sphere S^2 that has one loop-edge e and one vertex p, where both ends of e connect to p, and also includes a single closed loop γ_1 disjoint from e and p. Then equality holds in Equation (4), even though one of the components of $S^2 \setminus \mathcal{G}$ is homeomorphic to an annulus.

However, if $\mathcal{G}$ does not contain any closed loops, i.e. if $\mathcal{G} = \mathcal{G}'$, then equality holds in Equation (4) if and only if each component of $M \setminus \mathcal{G}$ is homeomorphic to a disk, as follows from the generalized Euler formula (Equation (6) below).

Proof. As $\mathcal{G}$ may contain closed loops, it is not a graph in the usual sense, so we cannot immediately apply the generalized Euler formula. We will add edges and vertices to $\mathcal{G}$ until it becomes a graph in the usual sense, and then apply the formula.

Let $V_1, ..., V_{\mathcal{F}}$ be the faces of $\mathcal{G}$ as in part 2 of Definition 2. Let $\mathcal{R}$ be the union of open regions V_j such that the boundary ∂V_j has nonempty intersection with $\mathcal{G}'$. Note that $\mathcal{R}$ is not empty, because $\mathcal{G}'$ is not empty. If $\mathcal{R} = M \setminus \mathcal{G}'$, then $\mathcal{G}$ contains no closed loops (i.e. $r = 0$) and $\mathcal{G} = \mathcal{G}'$ is a graph in the standard sense. If $\mathcal{R} \neq M \setminus \mathcal{G}'$, then there must exist some j_0 such that $V_{j_0} \subseteq \mathcal{R}$ and ∂V_{j_0} contains a loop γ_{j_1} for some j_1. By reordering the γ_j if necessary, we may assume $j_1 = r$. Since V_{j_0} has boundary components in both $\mathcal{G}'$ and γ_r, it is not simply-connected and we can add a vertex p at any place along γ_r and connect p by an edge e to some vertex of $\mathcal{G}'$ so that the interior of e lies in V_{j_0} and e does not disconnect V_{j_0}. Including p and e results in a graph-with-loops

$$\mathcal{G}_{r-1} = \mathcal{G}'_{r-1} \cup \sum_{j=1}^{r-1} \gamma_j \ , \qquad \text{where} \quad \mathcal{G}'_{r-1} = \mathcal{G}' \cup \{e\} \cup \{p\} \cup \{\gamma_r \setminus \{p\}\}$$

and the closed loop γ_r has become the loop-edge $\gamma_r \setminus \{p\}$ in $\mathcal{G}'_{r-1}$. In particular, $\mathcal{G} \subseteq \mathcal{G}_{r-1}$ (as sets in M). (The subscript $r-1$ in $\mathcal{G}_{r-1}$ indicates that $\mathcal{G}_{r-1}$ has $r-1$ closed loops.)

Denoting by $\mathcal{F}_{r-1}$, $\mathcal{E}_{r-1}$ and $\mathcal{V}_{r-1}$ the number of faces, edges and vertices of $\mathcal{G}_{r-1}$, we have that $\mathcal{F}_{r-1} = \mathcal{F}$, $\mathcal{E}_{r-1} = \mathcal{E} + 1$ and $\mathcal{V}_{r-1} = \mathcal{V} + 1$, hence

$$\mathcal{F}_{r-1} - \mathcal{E}_{r-1} + \mathcal{V}_{r-1} = \mathcal{F} - \mathcal{E} + \mathcal{V} \ .$$

Repeating this procedure $r-1$ more times, we can make a sequence of graphs-with-loops $\mathcal{G}_{r-1}$, $\mathcal{G}_{r-2}$, ..., $\mathcal{G}_0$ that change all the closed loops γ_j one by one into loop-edges. Each $\mathcal{G}_s$ has s closed loops, and

$\mathcal{G}_s \subseteq \mathcal{G}_t$ (as sets in M) when $t \leq s$. The final graph-with-loops $\mathcal{G}_0$ has no closed loops and hence is actually a graph in the standard sense, i.e. $\mathcal{G}_0 = \mathcal{G}_0'$. We have $\mathcal{G} \subseteq \mathcal{G}_0$ (as sets in M), and the number of faces $\mathcal{F}_0$ of $\mathcal{G}_0$ equals $\mathcal{F}$. Furthermore, defining $\mathcal{E}_0$ and $\mathcal{V}_0$ as the number of edges and vertices of $\mathcal{G}_0$, we have

$$\mathcal{F}_0 - \mathcal{E}_0 + \mathcal{V}_0 = \mathcal{F} - \mathcal{E} + \mathcal{V} .$$

So it is sufficient to show that

$$\mathcal{F}_0 - \mathcal{E}_0 + \mathcal{V}_0 \geq \chi(M) . \tag{5}$$

The graph $\mathcal{G}_0$ will have loop-edges if $r \geq 1$, but it has no closed loops, so the generalized Euler formula (see, for example, Chapter 9 of [12]) can be applied to $\mathcal{G}_0$. Letting $V_{1,0}, ..., V_{\mathcal{F},0}$ be the faces of $\mathcal{G}_0$, define $\chi(V_{j,0})$ to be the Euler characteristic of $V_{j,0}$. ($\chi(V_{j,0})$ can be computed using any true triangulation of $V_{j,0}$.) The generalized Euler formula says that

$$\mathcal{V}_0 - \mathcal{E}_0 + \sum_{j=1}^{\mathcal{F}} \chi(V_{j,0}) = \chi(M) . \tag{6}$$

Since $\chi(V_{j,0}) \leq 1$, this implies Equation (5). □

The next lemma is the Courant nodal domain theorem. The proof is well known (see [9] or [8], for example), but we include it here because we add a potential function to the Laplacian operator (this has little effect on the proof), and also because we consider the case of multiple eigenvalues.

Let M be a closed Riemann surface with smooth metric ds^2. Let dA and ∇ and $\triangle$ be the area form and gradient and Laplace-Beltrami operator on M associated to ds^2. We choose the sign of $\triangle$ so that $\int_M \phi \triangle \phi dA = + \int_M |\nabla \phi|^2 dA$ for general smooth functions ϕ on M. Consider the operator

$$c \cdot \triangle + V \tag{7}$$

on M, where c is a positive constant and V is a smooth bounded function on M. We write the complete set of eigenvalues for $c \cdot \triangle + V$ as

$$\lambda_1 < \lambda_2 \leq \lambda_3 \leq ... \nearrow +\infty$$

with associated smooth eigenfunctions ν_j, i.e.

$$(c \cdot \triangle + V)\nu_j = \lambda_j \nu_j ,$$

where the ν_j are chosen to form an orthonormal basis for the function space $L^2(M)$ on M.

The Sobolev space $H^1(M)$ of M is defined to be functions in $L^2(M)$ whose weak first derivatives exist and are also in $L^2(M)$. (Here the L^2 norm is defined with respect to the metric ds^2 on M.) There is a standard H^1 norm, with respect to ds^2, which makes $H^1(M)$ a Hilbert space.

Definition 3. For a function $\nu : M \to \boldsymbol{R}$, the set $\nu^{-1}(0)$ is the *nodal set* of ν, and each component of $M \setminus \nu^{-1}(0)$ is a *nodal domain* of ν.

Lemma 3. *The number of nodal domains of ν_j is at most j, for every $j \in \boldsymbol{Z}^+$. Furthermore, in the case of a multiple eigenvalue $\lambda_i = \lambda_{i+1} = ... = \lambda_{i+k}$, the number of nodal domains of any eigenfunction $\nu \in span\{\nu_i, ..., \nu_{i+k}\}$ is at most i.*

Proof. Assume ν_j has at least $j+1$ nodal domains, $j+1$ of which are $\Omega_1, ..., \Omega_{j+1}$. Define the function ϕ_k by

$$\phi_k = \nu_j \text{ on } \Omega_k , \quad \phi_k = 0 \text{ elsewhere,}$$

for $k = 1, ..., j$ (we exclude $k = j+1$). By Theorems 2.2 and 2.5 in [9], the boundary of each Ω_k is piece-wise smooth, and consists of a finite number of smooth curves of finite length and finite total curvature, so the weak first derivatives of ϕ_k exist and are bounded. Hence $\phi_k \in H^1(M)$. Clearly, $\langle \phi_{k_1}, \phi_{k_2} \rangle_{L^2} = 0$ for any unequal k_1 and k_2, so span$\{\phi_1, ..., \phi_j\}$ is of dimension j. Then, since span$\{\nu_1, ..., \nu_{j-1}\}$ is of dimension $j-1$, there exists some linear combination

$$\phi = \sum_{k=1}^{j} a_k \phi_k , \quad a_k \in \boldsymbol{R}$$

that is L^2-perpendicular to $\text{span}\{\nu_1, ..., \nu_{j-1}\}$, i.e.

$$\phi \in (\text{span}\{\nu_1, ..., \nu_{j-1}\})^{\perp_{L^2}} \quad . \tag{8}$$

Furthermore,

$$\phi \in H^1(M) \; . \tag{9}$$

Because ϕ is a linear combination of the ϕ_k, the Rayleigh quotient $R(\phi)$ of ϕ for the operator $c \cdot \triangle + V$ satisfies

$$R(\phi) := \frac{\int_M \phi((c \cdot \triangle + V)\phi) dA}{\int_M \phi^2 dA} = \lambda_j \; . \tag{10}$$

By (8), (9) and (10), it follows that ϕ is an eigenfunction with eigenvalue λ_j. (For arguments that show this, see [2], [5], [20] or [24], for example. In [5], the argument is given in full detail for the Laplacian operator on compact Riemannian manifolds, and the same argument can be applied to the operator $c \cdot \triangle + V$ here.)

However, the eigenfunction ϕ is identically zero on Ω_{k+1}. This contradicts the maximum principle (see [19], for example), and proves the first sentence of the lemma.

Now suppose that $\lambda_i = \lambda_{i+1} = ... = \lambda_{i+k}$ and that ν is any function in $\text{span}\{\nu_i, ..., \nu_{i+k}\}$. We are free to choose the functions $\nu_1, \nu_2, \nu_3, ...$ so that $\nu/||\nu||_{L^2} = \nu_i$, and then the argument in the previous two paragraphs shows that ν has at most i nodal domains. Since $\nu \in \text{span}\{\nu_i, ..., \nu_{i+k}\}$ is arbitrary, the second sentence of the lemma is shown. □

5. Linear lower bounds for the Morse index of the tori

Theorem 4. *If the torus F has spectral genus g and $m \geq 1$ disjoint congruent open pieces representing regions of double periodicity for u, then*

$$Ind(F) \geq m \cdot \left[\frac{g-1}{3}\right] - 2 \; ,$$

where $[r]$ denotes the greatest integer less than or equal to a real number r.

Proof. The theorem is vacuously true if $g \leq 3$, so we may assume $g \geq 4$. Let the v_j be as in Equations (1) and (2).

Since $\{\sum_{j=1}^{g-1} a_j v_j \,|\, a_j \in \boldsymbol{R}\}$ is a $g-1$ dimensional space in the null-space of the operator L, we can choose a_j so that $v = \sum_{j=1}^{g-1} a_j v_j$ has zeroes of order 1 at $\left[\frac{g-1}{3}\right]$ arbitrary distinct points $p_j \in \boldsymbol{C}/\widetilde{\Lambda}$ for $j = 1, ..., \left[\frac{g-1}{3}\right]$; that is, at each of these points p_j we have

$$v(p_j) = \partial_x v(p_j) = \partial_y v(p_j) = 0 \; .$$

Thus at each of these points p_j the nodal set (zero set) of v is locally a crossing with equiangular intersection of at least two curves, by Theorem 2.5 of [9].

Since we may consider the v_j to be functions on $\boldsymbol{C}/\Lambda$, we now consider v to be a function defined on $\boldsymbol{C}/\Lambda$. Let $\mathcal{G}$ be the nodal set of v on $\boldsymbol{C}/\Lambda$. Theorems 2.2 and 2.5 in [9] imply that $\mathcal{G}$ forms a graph-with-loops on $\boldsymbol{C}/\Lambda$ with smooth edges and isolated vertices where an even number of edges meet equiangularly. We also note that

$$\#(\text{vertices of } \mathcal{G}) \geq m \cdot \left[\frac{g-1}{3}\right] \geq 1 \; . \tag{11}$$

Lemma 2 and the first remark following it imply

$$\#(\text{components of } (\boldsymbol{C}/\Lambda) \setminus \mathcal{G}) \geq \#(\text{edges of } \mathcal{G}) - \#(\text{vertices of } \mathcal{G}) \; . \tag{12}$$

Since each vertex of $\mathcal{G}$ has degree at least four, we have the inequality

$$\#(\text{edges of } \mathcal{G}) \geq 2(\#(\text{vertices of } \mathcal{G})) \; . \tag{13}$$

Equations (11), (12) and (13) combine to give

$$\#(\text{components of } (\boldsymbol{C}/\Lambda) \setminus \mathcal{G}) \geq m \cdot \left[\frac{g-1}{3}\right] . \tag{14}$$

Let λ_j and ν_j be the eigenvalues and eigenfunctions (as defined in Section 3) of L. Since L has $\mathcal{K}$ negative eigenvalues, and 0 is a multiple eigenvalue of order at least $g-1$, we have $\lambda_{\mathcal{K}} < 0$ and $0 = \lambda_{\mathcal{K}+1} = ... = \lambda_{\mathcal{K}+k}$ and $0 < \lambda_{\mathcal{K}+k+1}$ for some $k \geq g-1$. So we have

$$v \in \text{span}\{v_1, ..., v_{g-1}\} \subseteq \text{span}\{\nu_{\mathcal{K}+1}, ..., \nu_{\mathcal{K}+k}\} .$$

Consider $\boldsymbol{C}/\Lambda$ with the Euclidean metric ds^2_{Eucl}, then the associated Laplace-Beltrami operator is $\triangle_{Eucl} = -\partial_x\partial_x - \partial_y\partial_y$. Furthermore, $L = (1/4)\cdot\triangle_{Eucl} - \cosh u$ is of the form in (7), and then Lemma 3 implies that v has at most $\mathcal{K}+1$ nodal domains, so

$$\mathcal{K} - 1 \geq \#(\text{components of } (\boldsymbol{C}/\Lambda) \setminus \mathcal{G}) - 2 . \tag{15}$$

Combining Equations (14) and (15) with Lemma 1, we have

$$\text{Ind}(F) \geq \mathcal{K} - 1 \geq \#(\text{components of } (\boldsymbol{C}/\Lambda) \setminus \mathcal{G}) - 2 \geq m \cdot \left[\frac{g-1}{3}\right] - 2 .$$

□

This result can be improved if $i\text{Re}(Uz_0) + D = (0, 0, ..., 0) = \vec{0}$ for some z_0. In this case we can translate the parameter $z \to z + z_0$ and assume that $D = \vec{0}$. The theta function satisfies the symmetry $\theta(\omega) = \theta(-\omega)$, so when $D = \vec{0}$, it follows that $u(z) = u(-z)$. Let $0, w_1, w_2, w_3$ be the four distinct points in $\boldsymbol{C}/\widetilde{\Lambda}$ such that $2 \cdot w_l$ is contained in the lattice $\widetilde{\Lambda}$ for $l = 1, 2, 3$. The fact that $u(z) = u(-z)$ implies that also $u(w_l + z) = u(w_l - z)$ for any of the three half-periods $w_l, l = 1, 2, 3$. From the recursions defining v_j and leading up to (2), we have

$$v_j(z) = -v_j(-z) \text{ and } v_j(w_l + z) = -v_j(w_l - z) . \tag{16}$$

Theorem 5. *With the same conditions as in Theorem 4, if we also have $D = \vec{0}$, then*

$$\mathit{Ind}(F) \geq m \cdot \left(\left[\frac{g-1}{3}\right] + \left[\min(\frac{g-1}{3}, 4)\right]\right) - 2 .$$

Proof. As in the proof of Theorem 4, we can choose the $p_j \in \boldsymbol{C}/\widetilde{\Lambda}$ arbitrarily. Furthermore, we define $v = \sum_{j=1}^{g-1} a_j v_j$ just as in that proof, so that $Lv = 0$ and v has zeroes of order 1 at the p_j for $j = 1, ..., [\frac{g-1}{3}]$.

Choosing the first four points p_j to be $p_1 = 0$, $p_2 = w_1$, $p_3 = w_2$, $p_4 = w_3$, then the antisymmetry (16) implies $v(p_j + z) = -v(p_j - z)$ for $j \leq 4$, hence the nodal set of v locally has 2ℓ edges intersecting at p_j with ℓ odd - in particular, there are at least six edges intersecting at p_j, for $j \leq 4$. The result then follows exactly as in the proof of Theorem 4, simply by noting that in this case one can add

$$m \cdot \left[\min(\frac{g-1}{3}, 4)\right]$$

to the right-hand side of Equation (13) and this equation will still hold (because the p_j for $j \leq 4$ each have at least six adjacent edges). □

6. A quadratic lower bound for the Morse index of the tori

We conclude with another method for finding lower bounds for the index of closed CMC tori. This second method gives estimates that are weaker by many orders of magnitude for smaller values of g, but it has the advantage that its estimates grow quadratically in g.

Let M be a closed Riemann surface of genus G with smooth metric ds^2 that is conformal to the complex structure of M. Let dA and $\triangle$ be the area form and Laplace-Beltrami operator on M with respect to ds^2, with the same sign convention for $\triangle$ as in Section 4. Then, let

$$\beta_1 < \beta_2 \leq \beta_3 \leq ...$$

be the complete set of eigenvalues of $\triangle$. (Each β_k is repeated the number of times equal to its multiplicity.) Let

$$A = \int_M dA$$

be the area of M. Theorem 0.5 in [13] tells us that there exists a universal constant $\tilde{C} > 0$ so that for all $k \geq 1$,

$$\beta_k \leq \tilde{C}(G+1)\frac{k}{A} \; . \tag{17}$$

We wish to apply Equation (17) to the CMC 1 isometric immersions $F : \boldsymbol{C}/\Lambda \to \boldsymbol{R}^3$ in Section 2, with any spectral genus $g \geq 2$. So we take $M = \boldsymbol{C}/\Lambda$ and hence $G = 1$, and we take $ds^2 = e^u \cdot ds^2_{Eucl}$ on $\boldsymbol{C}/\Lambda$. Let K and $H = 1$ be the Gauss and mean curvatures of $F(\boldsymbol{C}/\Lambda)$, considered as functions on $\boldsymbol{C}/\Lambda$.

We have the following variational characterizations for the k'th eigenvalues $\beta_k - 2$ and $\hat{\beta}_k$ of the operators $\triangle - 2$ and $\triangle - 4H^2 + 2K$:

$$\beta_k - 2 = \inf_{M_k}\left(\sup_{\psi \in M_k, \psi \neq 0} \frac{\int_{\boldsymbol{C}/\Lambda} \psi((\triangle - 2)\psi) dA}{\int_{\boldsymbol{C}/\Lambda} \psi^2 dA}\right) \; , \tag{18}$$

$$\hat{\beta}_k = \inf_{M_k}\left(\sup_{\psi \in M_k, \psi \neq 0} \frac{\int_{\boldsymbol{C}/\Lambda} \psi((\triangle - 4H^2 + 2K)\psi) dA}{\int_{\boldsymbol{C}/\Lambda} \psi^2 dA}\right) \; , \tag{19}$$

where M_k runs through all k dimensional subspaces of $C^\infty(M)$. (For arguments that show this, see [2], [5] or [24], for example. Again we remark that the argument is given in full detail for the Laplacian operator in [5], and that same argument can be applied to the operators here).

Noting that $-4H^2 + 2K = 2K - 4 \leq -2$, the variational characterizations (18) and (19) imply

$$\beta_k - 2 \geq \hat{\beta}_k \; . \tag{20}$$

For the operator L, as in Equation (1), and for the eigenvalues λ_k of L, as in Section 3, we have the following variational characterization:

$$\lambda_k = \inf_{M_k}\left(\sup_{\psi \in M_k, \psi \neq 0} \frac{\int_{\boldsymbol{C}/\Lambda} \psi L \psi dxdy}{\int_{\boldsymbol{C}/\Lambda} \psi^2 dxdy}\right) \; .$$

Then, since the final equality of Equation (3) holds for any smooth function v, we have

$$\lambda_k = \inf_{M_k}\left(\sup_{\psi \in M_k, \psi \neq 0} \frac{\int_{\boldsymbol{C}/\Lambda} \psi((\triangle - 4H^2 + 2K)\psi) dA}{4\int_{\boldsymbol{C}/\Lambda} \psi^2 dxdy}\right) \; , \tag{21}$$

Since the respective forms dA and $dxdy$ differ by a positive factor e^u bounded away from both 0 and ∞, the variational characterizations (19) and (21) imply that the k'th eigenvalue $\hat{\beta}_k$ is negative if and only if the k'th eigenvalue λ_k is negative. Hence, by Lemma 1, $\mathrm{Ind}(F)$ is greater than or equal to one less than the number of negative eigenvalues $\hat{\beta}_k$ of the operator $\triangle - 4H^2 + 2K$. Then, with

$$A = \mathrm{area}(F(\boldsymbol{C}/\Lambda)) \; ,$$

the inequalities (17) and (20) imply

$$\mathrm{Ind}(F) \geq \#\{k \mid \beta_k < 2\} - 1 \geq \left[\frac{2A}{\tilde{C}(G+1)}\right] - 2 = \left[\frac{A}{\tilde{C}}\right] - 2 \; . \tag{22}$$

When the CMC 1 torus $F(\boldsymbol{C}/\Lambda)$ has spectral genus g, it is shown in [11] that

$$A \geq \frac{\pi}{4}\left((g+2)^2 - \frac{1}{2}(1 + (-1)^g)\right) \; . \tag{23}$$

Combining inequalities (22) and (23) results in:

Theorem 6. *There exists a universal constant $C > 0$ such that if the torus F has spectral genus g, then*
$$Ind(F) \geq C\left((g+2)^2 - \frac{1}{2}(1+(-1)^g)\right) - 2 \geq Cg^2 - 2 \,.$$

Because $\tilde{C}$ in [13] is on the order of 10^7, C is on the order of 10^{-7}, so even though the lower bound in Theorem 6 grows quadratically in g, it is much weaker than the lower bounds in Theorems 4 and 5 for smaller values of g.

Remark. One could make similar arguments using Theorem 16 of [14] instead of Theorem 0.5 in [13], but then one has a lower bound that also depends on the diameter and a lower bound for the Gaussian curvature of the surface.

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